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Double spectral representations of single loop amplitudes with k vertices: $k \geq 4$

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A method developed in several previous papers is combined with the method of induction to derive double dispersion relations, with Mandelstam boundary, for the class of single loop amplitudes with four or more vertices. The spectral functions are expressed as integral representations and restrictions on the masses and kinematic invariants for which dispersion relations are valid are found. It is also discussed how representations for the low order single loop amplitudes can be obtained for wider ranges of these variables.

1. INTRODUCTION

Since the late 1950's, when it became apparent that dispersion relations for nuclen nucleon scattering and the nucleon electromagnetic form factor could not be proved on the basis of the general principles of field theory,[1] there has been a flood of literature on the analytic properties of Feynman amplitudes.[2-4] Because of their relative simplicity, special attention was initially devoted to the study of the low order single loop amplitudes in ϕ^3 theory. These investigations led to the introduction of some important new concepts. In particular Karplus, Sommerfield, and Wichmann,[5] Nambu,[6] and Oehme[7] discovered the anomalous threshold of the triangle diagram vertex function and Mandelstam[8] showed that, for a restricted range of masses, the box diagram amplitude satisfies the famous double spectral representation that bears his name.[9] The single loop diagrams have also played a central role in the majorization procedure,[2-4] which is aimed at showing that all Feynman amplitudes contributing to a particular process involving a given number of external particles are regular functions in a domain whose extent is determined by one or more of the simple diagrams.

With the advent of the Landau–Cutkosky rules[10,11] it became possible, in principle, to determine the singularities of a general Feynman integral and as well to obtain the discontinuities across the corresponding branch cuts. While these rules have been enormously useful in studying the analytic properties of Feynman amplitudes[2-4] and for obtaining heuristic dispersion relations for certain processes,[12,13] they are, by themselves, not sufficient for a rigorous derivation of dispersion relations. One of the main problems is that they do not determine on which Riemann sheets the singularities lie,[13-15] and in particular which singularities lie on the physical sheet. Further the discontinuity can in general only be determined up to a sign factor.

To overcome these problems, Fotiadi, Froissart, Lascoux, and Pham[16] proposed in 1963 that homology theory be used as a rigorous way of studying the analytic properties of individual Feynman integrals. Again, the investigations made using this method have been mainly restricted to the single loop diagrams and especially the low order single loop diagrams,[17] since the application of homology theory to more complicated diagrams has proved to be much more difficult.[18]

It is the aim of this paper to show that, within ϕ^3 theory, double dispersions relation with Mandelstam boundary can, for a restricted range of masses and kinematic invariants, be proved for any Feynman amplitude arising from a single loop diagram with four or more vertices. Further we obtain integral representations for the weight functions and discuss the significance of the above restrictions on the masses and kinematic invariants. The method used to derive these results is a combination of a method developed in several previous papers[19-21] with the method of induction; it involves the direct transformation of the Feynman parametrized form of the kth order single loop amplitude ($k \geqslant 4$) into the required form. (Refs. 19, 20, 21, are referred to as VF, I and P respectively.)

In Sec. 2 the first of the two Cauchy kernels needed for the double dispersion relation is introduced by changing the variables in the Feynman parametrized form of the kth order single loop amplitude. The restrictions on the masses and kinematic invariants for which this new form of the amplitude is valid are also discussed in this section. The boundary of the region of integration in the multiple integral representation derived in Sec. 2 is studied in Sec. 3, and in Sec. 4 we obtain some results necessary for reversing the orders of integration.

In Sec. 5, the orders of some of the integrations are reversed and the second Cauchy kernel is introduced by changing one of the variables of integration. The boundary of the region of integration in the resultant new multiple integral representation is studied in Sec. 6 and in Sec. 7 further results necessary for the reversal of the orders of integration are obtained.

Finally in Sec. 8 the required double dispersion relation for the kth order single loop amplitude is derived by changing the orders of integration in the integral representation obtained in Sec. 5. The spectral function is expressed in the form of a multiple integral, and it is found that the boundary of the double spectral representation is the usual Mandelstam boundary for the box diagram amplitude. In this section we also discuss in detail the implications of the restrictions on the masses and kinematic invariants made in Sec. 2 and how these restrictions may to a certain extent be relaxed.

2. TRANSFORMATION OF k^{th} ORDER SINGLE LOOP AMPLITUDE: $k \geqslant 4$

With plane wave states normalized, so that $\langle \mathbf{p}' | \mathbf{p} \rangle = \delta^3(\mathbf{p}' - \mathbf{p})$, we define the scalar invariant amplitude T_k for the multiparticle production process in which i initial particles produce $f = (k - i)$ final particles by

$$\langle \mathbf{p}_1, \ldots, \mathbf{p}_l | S-1 | -\mathbf{p}_{l+1}, \ldots, -\mathbf{p}_k \rangle = -i(2\pi)^4 \delta^{(4)}\left(\sum_{i=1}^{k} p_i\right)(2\pi)^{-3k/2}2^{-k/2}\left(\prod_{i=1}^{k} E_i\right)^{-1/2} T_k. \quad (1)$$

Our object is to show that the contribution to T_k from the kth order single-loop diagram shown in Fig. 1 or equivalently in Fig. 2 can, for a restricted range of masses and kinematic invariants, be written as a double spectral representation with Mandelstam boundary. Further we shall obtain an integral representation for the spectral function.

The k external momenta shown in Figs. 1 and 2 are labeled by the subscripts of the adjacent internal masses and the external mass squared of a particular external line is, of course, just the square of the external momentum of that line. The other variables on which the single loop amplitude depends are most conveniently defined in terms of the external momenta in Fig. 1 by

$$q_{ij}^2 = \left(\sum_{r=i}^{j-1} q_{r,r+1}\right)^2 \quad (1 \le i < j \le k). \quad (2)$$

It should be noted that when $k \ge 6$ the above kinematic invariants are not independent but satisfy algebraic constraints.[22] Notice also that when $j = i+1$ so that q_{ij}^2 is an external mass squared, Eq. (2) becomes an identity. Further, with the powers 2 removed, Eq. (2) is just the energy momentum conservation law when $i=1$, $j=k$ (since $-q_{1k}$ rather than q_{1k} is the ingoing 4-momentum in Fig. 1).

We shall find, however, that many of the subsequent expressions needed to obtain the double spectral representations take a simpler form in terms of the asymmetrically labelled variables shown in Fig. 2. The relations between the two sets of variables can be seen from Figs. 1 and 2. With $n = k-2$, they are as follows:

$$m_{-1} = M_1, \quad m_0 = M_{(k-1)}, \quad m_1 = M_k,$$
$$p_{-10} = q_{1(k-1)}, \quad p_{-11} = q_{1k}, \quad p_{01} = q_{(k-1)k},$$
$$m_j = M_j, \quad p_{-1j} = q_{1j}, \quad p_{1j} = q_{jk}, \quad p_{0j} = q_{j(k-1)},$$
$$(2 \le j \le k-2 = n)$$
$$p_{ij} = q_{ij} \quad (2 \le i < j \le k-2 = n), \quad (3)$$

where the q_{ij} are given in Eq. (2). We now define

$$y_{ij} = -(2m_i m_j)^{-1}[p_{ij}^2 - m_i^2 - m_j^2] \quad (-1 \le i < j \le n)$$
$$y_{ij} = y_{ji}, \quad y_{ii} = 1 \quad (4)$$

and as well

$$x_1 \equiv -y_{-10}, \quad x_2 \equiv -y_{12}. \quad (5)$$

We shall write the dispersion relations in x_1 and x_2 which are linearly related to the usual Mandelstam variables s and t.

Then, using standard Feynman rules[23] (see also Ref. 24 and Section 1.5 of Ref. 2) we find that the amplitude arising from the kth order single loop diagram shown in Fig. 2 takes the form

$$T_{(n+2)\,\text{loop}}(y_{ij}) = \frac{g}{16\pi^2} \cdot \frac{(-1)^n}{2m_{-1}m_0m_1m_2(n-1)!} I_{n+2}(y_{ij}), \quad (6)$$

where

$$I_{n+2}(y_{ij}) = 2m_{-1}m_0m_1m_2(-1)^n(n-1)! \times \int_{R_n} \prod_{\substack{-1\\ i\ne 1}}^{n} d\alpha_i [D_n(\alpha_{-1}, \alpha_0, \alpha_2, \ldots, \alpha_n)]^{-n}, \quad (7)$$

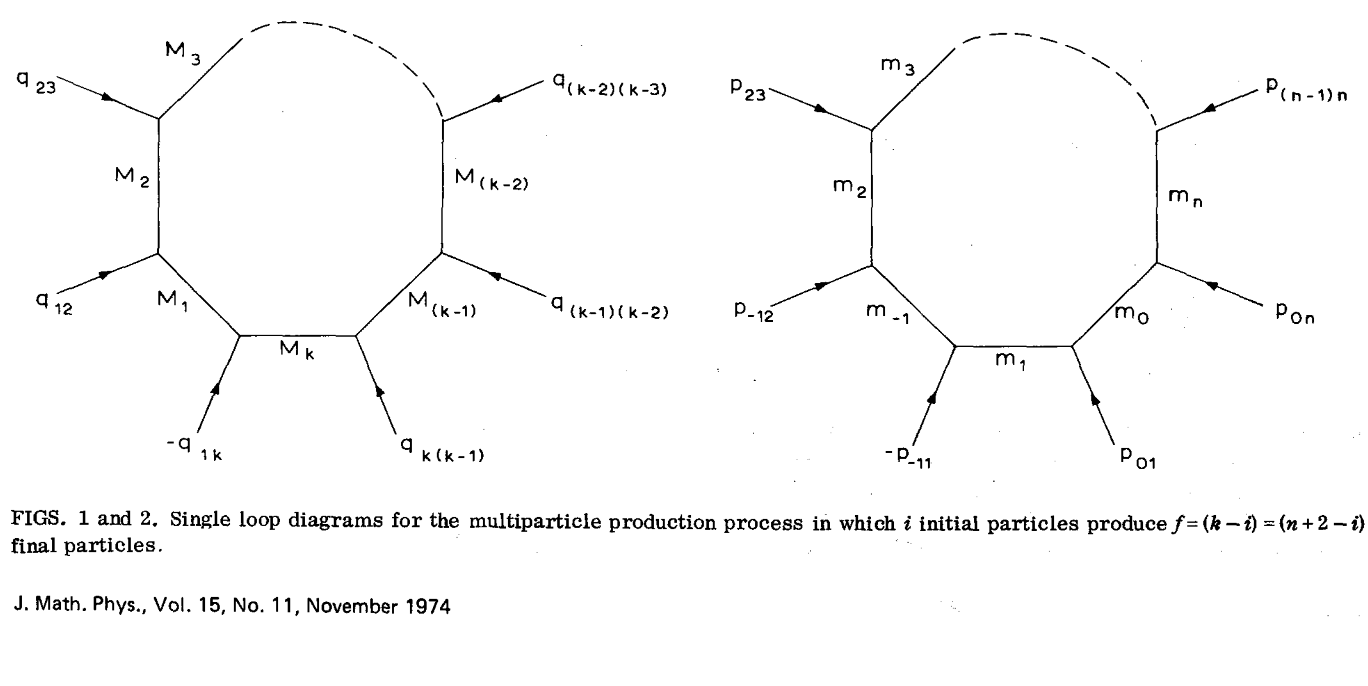
$$D_n(\alpha_{-1}, \alpha_0, \alpha_2, \ldots, \alpha_n) = \sum_{\substack{-1\\ i\ne 1}}^{n} m_i^2\alpha_i^2 + m_1^2\Big(1 - \sum_{\substack{-1\\ i\ne 1}}^{n}\alpha_i\Big)^2 + \sum_{\substack{i<j\\ i,j\ne 1}}^{n} 2m_im_jy_{ij}\alpha_i\alpha_j + \sum_{\substack{-1\\ j\ne 1}}^{n} 2m_1m_jy_{1j}\alpha_j\Big(1 - \sum_{\substack{-1\\ i\ne 1}}^{n}\alpha_i\Big) \quad (8)$$

and

$$R_n = \Big\{(\alpha_{-1}, \alpha_0, \alpha_2, \ldots, \alpha_n) \,\Big|\, \alpha_{-1} \ge 0,\ \alpha_0 \ge 0,\ \alpha_2 \ge 0, \ldots, \alpha_n \ge 0,\ \sum_{\substack{-1\\ i\ne 1}}^{n}\alpha_i = 1\Big\}. \quad (9)$$

The constant g is the product of the coupling constants acting at the vertices in Fig. 2.

We begin by generalizing the transformation used in Sec. 2 of P. The change of variables is



FIGS. 1 and 2. Single loop diagrams for the multiparticle production process in which i initial particles produce $f = (k-i) = (n+2-i)$ final particles.

$$\nu = \alpha_0^{-1}(\alpha_{-1}+\alpha_0),$$
$$\lambda_1 = (\alpha_{-1}+\alpha_0)^{-1}\Big(1-\sum_{\substack{-1\\ i\neq 1}}^{n}\alpha_i\Big),$$
$$\lambda_i = (\alpha_{-1}+\alpha_0)^{-1}\alpha_i \quad (2\leqslant i\leqslant n), \tag{10}$$

and it is shown by induction in Appendix A that the inverse is

$$\alpha_{-1} = \nu^{-1}(\nu-1)\Big(1+\sum_{i=1}^{n}\lambda_i\Big)^{-1},$$
$$\alpha_0 = \nu^{-1}\Big(1+\sum_{i=1}^{n}\lambda_i\Big)^{-1},$$
$$\alpha_i = \lambda_i\Big(1+\sum_{i=1}^{n}\lambda_i\Big)^{-1} \quad (2\leqslant i\leqslant n). \tag{11}$$

Further, from Appendix A, we find that the Jacobian of the transformation is

$$\left|\frac{\partial(\alpha_{-1},\alpha_0,\alpha_2,\ldots,\alpha_n)}{\partial(\nu,\lambda_1,\ldots,\lambda_n)}\right| = \Big(1+\sum_{i=1}^{n}\lambda_i\Big)^{-(n+2)}\nu^{-2} \tag{12}$$

and

$$D_n(\alpha_{-1},\alpha_0,\alpha_2,\ldots,\alpha_n)\equiv\Delta_n(\nu,\lambda_1,\ldots,\lambda_n)$$
$$=\nu^{-1}\Big(1+\sum_{i=1}^{n}\lambda_i\Big)^{-2}[(\nu-1)\phi(\lambda_1,\ldots,\lambda_n)+\psi(\lambda_1,\ldots,\lambda_n)$$
$$-\nu^{-1}(\nu-1)v(x_1)]. \tag{13}$$

Here

$$\phi(\lambda_1,\ldots,\lambda_n)=\sum_{i=1}^{n}m_i^2\lambda_i^2+\sum_{i<j}^{n}2m_im_jy_{ij}\lambda_i\lambda_j$$
$$+\sum_{i=1}^{n}2m_{-1}m_iy_{-1i}\lambda_i+m_{-1}^2, \tag{14}$$

$$\psi(\lambda_1,\ldots,\lambda_n)=\sum_{i=1}^{n}m_i^2\lambda_i^2+\sum_{i<j}^{n}2m_im_jy_{ij}\lambda_i\lambda_j$$
$$+\sum_{i=1}^{n}2m_0m_iy_{0i}\lambda_i+m_0^2, \tag{15}$$

and

$$v(x_1)=2m_{-1}m_0x_1^2+m_{-1}^2+m_0^2 \tag{16}$$

with x_1 given in Eq. (5). Now $I_{n+2}(y_{ij})$ takes the form

$$I_{n+2}(y_{ij})=2m_{-1}m_0m_1m_2(-1)^n(n-1)!\int_0^\infty\prod_{i=1}^{n}d\lambda_i\int_1^\infty d\nu$$
$$\times\nu^{-2}\Big(1+\sum_{i=1}^{n}\lambda_i\Big)^{n-2}$$
$$\times\{\nu^{-1}[(\nu-1)\phi(\lambda_1,\ldots,\lambda_n)+\psi(\lambda_1,\ldots,\lambda_n)$$
$$-\nu^{-1}(\nu-1)v(x_1)]\}^{-n} \tag{17a}$$
$$=2m_{-1}m_0m_1m_2\int_0^\infty\prod_{i=1}^{n}d\lambda_i\Big(\prod_{i=3}^{n}\lambda_i\Big)^{-1}\int_1^\infty d\nu$$
$$\times\Big(\prod_{i=3}^{n}\frac{\partial}{\partial m_i^2}\Big)[(\nu-1)\phi(\lambda_1,\ldots,\lambda_n)+\psi(\lambda_1,\ldots,\lambda_n)$$
$$-\nu^{-1}(\nu-1)v(x_1)]^{-2} \tag{17b}$$

for $n\geqslant 3$. That the expression for $I_{n+2}(y_{ij})$ in Eq. (17b) is, for $n\geqslant 3$, equivalent to that in Eq. (17a) can be seen by using Eq. (4) in Eqs. (14) and (15). Equation (17a) of course holds for all $n\geqslant 2$, but as the case $n=2$ was treated in detail in I, we shall concentrate on the case $n\geqslant 3$. Note also that the form of Eq. (17b) is similar to Eq. (P-5); in fact the structures of many of the subsequent equations will be similar to those in P. [Equations from P (resp. I) are denoted by placing a P- (resp. I-) in front of the equation number.]

To simplify the proof of the spectral representations, we restrict the y_{ij} defined in Eq. (4) to

$$y_{ij}>0 \quad (-1\leqslant i<j\leqslant n). \tag{18}$$

Equation (18) ensures that $\phi(\lambda_1,\ldots,\lambda_n)>0$, $\psi(\lambda_1,\ldots,\lambda_n)>0$ for $\lambda_1\geqslant 0,\ldots,\lambda_n\geqslant 0$; in fact the term in square brackets in Eq. (17b) is always positive and $I_{n+2}(y_{ij})$ is well defined. That the conditions in Eq. (18) can, for sufficiently large internal masses, be satisfied for finite physical values of the kinematic invariants and external masses is shown in Sec. 8. Equation (18) in fact gives sufficient conditions for the external masses to be stable. In general, however, for a physical single-loop amplitude corresponding to i initial particles producing $f=(k-i)$ final particles, some of the kinematic invariants defined in Eq. (3) can be positive and unbounded. Thus, for finite internal masses it is possible for some of the kinematic invariants to have physical values such that the corresponding y_{ij} are negative. However, it can be seen from Eqs. (17b), (14), (15), and (16) that when some of the y_{ij} are negative, a spectral representation for $I_{n+2}(y_{ij})$ cannot in general be proved by using real analysis only. To obtain a representation in such cases, for physical values of the invariants, one might then start with the double spectral representation derived in Sec. 8 [Eq. (67)] and attempt to do an analytic continuation in the required kinematic invariants using, for example, a generalization of the method of Ref. 25 (referred to as II). Such a procedure might be feasible for the pentagon diagram amplitude, at least for some specific processes,[26] but for a general kth order single-loop amplitude this method does not seem practical for obtaining a representation for all possible configurations involving physical invariants. Of course, some continuation, namely in x_1 and x_2, can easily be carried out since these variables appear only in the Cauchy kernels in Eq. (67). Further, as discussed in Sec. 8, Eq. (67) is expected to be valid under more general conditions [on the other variables defined in Eq. (4) as well as on x_1 and x_2] than those given in Eq. (18).

The argument leading to Eqs. (I-19) and (I-20) can now be used to show that

$$I_{n+2}(y_{ij})=\int_0^\infty\Big(\prod_{\substack{3\\ j\neq i}}^{n}d\lambda_j\Big)^{-1}\Big(\prod_{\substack{3\\ j\neq i}}^{n}\frac{\partial}{\partial m_j^2}\Big)\lim_{\epsilon_i\downarrow 0}\frac{\partial}{\partial m_i^2}\int_{\epsilon_i}^\infty\frac{d\lambda_i}{\lambda_i}$$
$$\times J_{n+2}(y_{ij},\lambda_3,\ldots,\lambda_n) \tag{19a}$$
$$=\prod_{i=3}^{n}\Big(\lim_{\epsilon_i\downarrow 0}\frac{\partial}{\partial m_i^2}\int_{\epsilon_i}^\infty\frac{d\lambda_i}{\lambda_i}\Big)J_{n+2}(y_{ij},\lambda_3,\ldots,\lambda_n), \tag{19b}$$

where

$$J_{n+2}(y_{ij},\lambda_3,\ldots,\lambda_n)$$
$$=\int_0^\infty\frac{d\lambda_2}{\lambda_2}\lim_{\epsilon_1\downarrow 0}\frac{\partial}{\partial x_2}\int_{\epsilon_1}^\infty\frac{d\lambda_1}{\lambda_1}\int_{h(\lambda_1,\ldots,\lambda_n)}^\infty$$
$$\frac{d\xi}{(\xi-x_1)[U(\xi,\lambda_1,\ldots,\lambda_n)]^{1/2}}. \tag{20}$$

In Eq. (20)

$$U(\xi, \lambda_1, \ldots, \lambda_n) = [\xi - h(\lambda_1, \ldots, \lambda_n)][\xi - k(\lambda_1, \ldots, \lambda_n)], \tag{21}$$

$$\begin{matrix} h(\lambda_1, \ldots, \lambda_n) \\ k(\lambda_1, \ldots, \lambda_n) \end{matrix} = (2m_{-1}m_0)^{-1}\{[\sqrt{\phi(\lambda_1, \ldots, \lambda_n)} \pm \sqrt{\psi(\lambda_1, \ldots, \lambda_n)}]^2 - m_{-1}^2 - m_0^2\} \tag{22}$$

and $\phi(\lambda_1, \ldots, \lambda_n)$, $\psi(\lambda_1, \ldots, \lambda_n)$ are given in Eqs. (14) and (15). The required Cauchy kernel now appears in Eqs. (19) and (20) and to obtain a dispersion relation the orders of integration must be reversed so that the lower limit of the ξ integration becomes a constant.

3. STUDY OF h (λ)

To reverse the order of integration in Eqs. (19) and (20), we need to examine the function $h(\lambda_1, \ldots, \lambda_n)$ for $\lambda_1 \geq 0, \ldots, \lambda_n \geq 0$. For convenience we introduce the following notation:

$$(\boldsymbol{\lambda}) \equiv (\lambda_1, \ldots, \lambda_n), \tag{23}$$

$$({}_{i\ldots l}\boldsymbol{\lambda}) \equiv (\lambda_1, \ldots, \lambda_{i-1}, 0, \lambda_{i+1}, \ldots, \lambda_{l-1}, 0, \lambda_{l+1}, \ldots, \lambda_n), \tag{24}$$

and

$${}_{i\ldots l}\boldsymbol{\lambda} \geq 0 \equiv \{\lambda_1 \geq 0, \ldots, \lambda_{i-1} \geq 0, \lambda_{i+1} \geq 0, \ldots, \lambda_{l-1} \geq 0, \lambda_{l+1} \geq 0, \ldots, \lambda_n \geq 0\} \tag{25}$$

where $i, \ldots, l \in \{1, \ldots, n\}$. Then from Eqs. (14) and (15),

$$\phi(\boldsymbol{\lambda}) = p_i\lambda_i^2 + 2q_i({}_i\boldsymbol{\lambda})\lambda_i + r_i({}_i\boldsymbol{\lambda}) \quad (>0),$$

$$\psi(\boldsymbol{\lambda}) = p_i\lambda_i^2 + 2q_i'({}_i\boldsymbol{\lambda})\lambda_i + r_i'({}_i\boldsymbol{\lambda}) \quad (>0), \tag{26}$$

where $i \in \{1, \ldots, n\}$ is fixed and

$$p_i = m_i^2,$$

$$q_i(\boldsymbol{\lambda}) = m_i\left(\sum_{j=1}^{n} m_j y_{ij}\lambda_j + m_{-1}y_{-1i}\right) \quad (>0),$$

$$q_i'(\boldsymbol{\lambda}) = m_i\left(\sum_{j=1}^{n} m_j y_{ij}\lambda_j + m_0 y_{0i}\right) \quad (>0),$$

$$r_i(\boldsymbol{\lambda}) \equiv \phi(\boldsymbol{\lambda}), \quad r_i'(\boldsymbol{\lambda}) \equiv \psi(\boldsymbol{\lambda}). \tag{27}$$

We have chosen to define $q_i(\boldsymbol{\lambda})$ etc. although only $q_i({}_i\boldsymbol{\lambda})$ etc. are needed in Eq. (26). The functions $r_i({}_i\boldsymbol{\lambda})$ and $r_i'({}_i\boldsymbol{\lambda})$ are determined recursively from Eq. (26) by putting λ_i equal to zero and using in addition Eq. (27) and the fact that $r_i(0) = m_{-1}^2$, $r_i'(0) = m_0^2$.

The argument of Sec. 4 of I (or of VF) then shows that for fixed ${}_i\boldsymbol{\lambda} \geq 0$, $h(\boldsymbol{\lambda})$ increases strictly from $h({}_i\boldsymbol{\lambda})$ to $+\infty$ as λ_i increases from 0 to $+\infty$, whenever $h_{\lambda_i}({}_i\boldsymbol{\lambda}) \geq 0$. Now

$$h_{\lambda_i}({}_i\boldsymbol{\lambda}) = (m_{-1}m_0)^{-1}[\sqrt{r_i({}_i\boldsymbol{\lambda})} + \sqrt{r_i'({}_i\boldsymbol{\lambda})}]l_i({}_i\boldsymbol{\lambda}), \tag{28}$$

where

$$l_i({}_i\boldsymbol{\lambda}) = \frac{q_i({}_i\boldsymbol{\lambda})}{\sqrt{r_i({}_i\boldsymbol{\lambda})}} + \frac{q_i'({}_i\boldsymbol{\lambda})}{\sqrt{r_i'({}_i\boldsymbol{\lambda})}} \quad (>0), \tag{29}$$

which is positive whenever Eq. (18) holds. Thus we have established that for fixed $i \in \{1, \ldots, n\}$ and fixed ${}_i\boldsymbol{\lambda} \geq 0$, $h(\boldsymbol{\lambda})$ increases strictly from $h({}_i\boldsymbol{\lambda})$ to $+\infty$ as λ_i increases from 0 to $+\infty$. Next we shall find the inverse of $\xi = h(\boldsymbol{\lambda})$ for fixed ${}_i\boldsymbol{\lambda} \geq 0$.

4. SOLUTIONS OF $U(\xi, \lambda) = 0$

In this section we study the behavior of the zeros of $U(\xi, \boldsymbol{\lambda})$ for fixed $\xi \geq h({}_i\boldsymbol{\lambda})$ and for fixed ${}_i\boldsymbol{\lambda} \geq 0$. From Eqs. (21), (22), (26), and (27) we have, for fixed $i \in \{1, \ldots, n\}$,

$$4m_{-1}^2m_0^2U(\xi, \boldsymbol{\lambda}) = a_i(\xi)\lambda_i^2 + 2b_i(\xi, {}_i\boldsymbol{\lambda})\lambda_i + c_i(\xi, {}_i\boldsymbol{\lambda}), \tag{30}$$

where

$$\begin{aligned} a_i(\xi) &= 4[(q_i(\boldsymbol{\lambda}) - q_i'(\boldsymbol{\lambda}))^2 - p_i v(\xi)] \\ &= 4m_i^2[(m_{-1}y_{-1i} - m_0 y_{0i})^2 - v(\xi)], \\ b_i(\xi, \boldsymbol{\lambda}) &= 2\{[q_i(\boldsymbol{\lambda}) - q_i'(\boldsymbol{\lambda})][r_i(\boldsymbol{\lambda}) - r_i'(\boldsymbol{\lambda})] - [q_i(\boldsymbol{\lambda}) + q_i'(\boldsymbol{\lambda})]v(\xi)\} \\ &= \sum_{j=1}^{n} \alpha_{ij}(\xi, -y_{ij})\lambda_j + b_i(\xi, \mathbf{0}), \\ \alpha_{ij}(\xi, -y_{ij}) &= 4m_i m_j[(m_{-1}y_{-1i} - m_0 y_{0i})(m_{-1}y_{-1j} - m_0 y_{0j}) - y_{ij}v(\xi)], \\ b_i(\xi, \mathbf{0}) &= -4m_i m_{-1}m_0[(m_{-1}y_{-1i} + m_0 y_{0i})\xi + m_{-1}y_{0i} + m_0 y_{-1i}] \end{aligned} \tag{31}$$

and $v(\xi)$ is given in Eq. (16). The functions $c_i(\xi, {}_i\boldsymbol{\lambda})$ are determined recursively from Eq. (30) by putting λ_i equal to zero and using in addition Eq. (31) and the fact that

$$c_i(\xi, \mathbf{0}) = 4m_{-1}^2m_0^2(\xi^2 - 1) \quad (1 \leq i \leq n). \tag{32}$$

The argument of Sec. 5 of VF (or I) shows that for each $\xi \geq h({}_i\boldsymbol{\lambda})$, where ${}_i\boldsymbol{\lambda} \geq 0$, the quadratic equation in λ_i,

$$U(\xi, \boldsymbol{\lambda}) = 0,$$

has two real roots given by

$$\lambda_{i\pm}(\xi, {}_i\lambda) = [a_i(\xi)]^{-1}\{-b_i(\xi, {}_i\boldsymbol{\lambda}) \mp [(b_i(\xi, {}_i\boldsymbol{\lambda}))^2 - a_i(\xi)c_i(\xi, {}_i\boldsymbol{\lambda})]^{1/2}\}. \tag{33}$$

From Eqs. (31), (22), (25), and (27) we see that

$$b_i(h({}_i\boldsymbol{\lambda}), {}_i\boldsymbol{\lambda}) = -4[\sqrt{r_i({}_i\boldsymbol{\lambda})} + \sqrt{r_i'({}_i\boldsymbol{\lambda})}]\sqrt{r_i({}_i\boldsymbol{\lambda})}\sqrt{r_i'({}_i\boldsymbol{\lambda})}l_i({}_i\boldsymbol{\lambda}), \tag{34}$$

where $l_i({}_i\boldsymbol{\lambda})$ is given in Eq. (29). Since $l_i({}_i\boldsymbol{\lambda}) > 0$ when Eq. (18) holds it follows that $\lambda_{i+}(h({}_i\boldsymbol{\lambda}), {}_i\boldsymbol{\lambda}) = 0 \neq \lambda_{i-}(h({}_i\boldsymbol{\lambda}), {}_i\boldsymbol{\lambda})$ and in fact for fixed ${}_i\boldsymbol{\lambda} \geq 0$ $\lambda_{i+}(\xi, {}_i\boldsymbol{\lambda})$ is the inverse of the strictly increasing function $h(\boldsymbol{\lambda})$ on $0 \leq \lambda_i < \infty$. Thus $\lambda_{i+}(\xi, {}_i\boldsymbol{\lambda})$ increases strictly from 0 to $+\infty$ as ξ increases from $h({}_i\boldsymbol{\lambda})$ to $+\infty$. We are now in a position to reverse the orders of the ξ and λ_i integrations $(1 \leq i \leq n)$ in Eqs. (19) and (20).

5. REVERSAL OF THE ORDER OF INTEGRATION

We begin this section by reversing the orders of the ξ and λ_1 integrations in Eq. (20). From Secs. 3 and 4 it follows in particular that $h_{\lambda_1}({}_1\boldsymbol{\lambda}) > 0$ for all ${}_1\boldsymbol{\lambda} \geq 0$ and that, for fixed ${}_1\boldsymbol{\lambda} \geq 0$, $\lambda_{1+}(\xi, {}_1\boldsymbol{\lambda})$ is the inverse of the strictly increasing function $h(\boldsymbol{\lambda})$ on $0 \leq \lambda_1 < \infty$. Thus

$$J_{n+2}(y_{ij},{}_{12}\boldsymbol{\lambda})=2m_{-1}m_0\int_0^\infty\frac{d\lambda_2}{\lambda_2}\lim_{\epsilon_1\downarrow 0}\frac{\partial}{\partial x_2}\times\int_{h(\epsilon_1,{}_1\lambda)}^\infty\frac{d\xi}{\xi-x_1}\cdot\Lambda(\xi,\epsilon_1,{}_1\boldsymbol{\lambda}) \tag{35}$$

where

$$\Lambda(\xi,\epsilon_1,{}_1\boldsymbol{\lambda})=\int_{\epsilon_1}^{\lambda_{1+}(\xi,{}_1\lambda)}\frac{d\lambda_1}{\lambda_1[a_1(\xi)\lambda_1^2+2b_1(\xi,{}_1\boldsymbol{\lambda})\lambda_1+c_1(\xi,{}_1\boldsymbol{\lambda})]^{1/2}}, \tag{36}$$

and $a_1(\xi)$, $b_1(\xi,{}_1\boldsymbol{\lambda})$, and $c_1(\xi,{}_1\boldsymbol{\lambda})$ can be obtained from Eqs. (30)–(32). Note that $h(\epsilon_1,{}_1\boldsymbol{\lambda})$, $b_1(\xi,{}_1\boldsymbol{\lambda})$, $\lambda_{1+}(\xi,{}_1\boldsymbol{\lambda})$, and $\Lambda(\xi,\epsilon_1,{}_1\boldsymbol{\lambda})$ depend on x_2, where x_2 is given in Eq. (5).

Now, since, for fixed ${}_{12}\boldsymbol{\lambda}\geq 0$, $\lambda_{2+}(\xi,{}_{12}\boldsymbol{\lambda})$ is the inverse of the strictly increasing function $h({}_1\boldsymbol{\lambda})$ on $0\leq\lambda_2<\infty$, the argument of Sec. 6 of I (or Sec. 5 of P) can be used to show that

$$J_{n+2}(y_{ij},{}_{12}\boldsymbol{\lambda})=\int_{h({}_{12}\lambda)}^\infty\frac{d\xi}{\xi-x_1}X(\xi,{}_{12}\boldsymbol{\lambda}), \tag{37}$$

where

$$X(\xi,{}_{12}\boldsymbol{\lambda})=8m_{-1}m_0m_1m_2v(\xi)\int_{f_+(\xi,{}_{12}\lambda)}^\infty\frac{d\eta}{(\eta-x_2)[\bar{F}(\xi,\eta,{}_{12}\boldsymbol{\lambda})]^{1/2}} \tag{38}$$

In Eq. (38)

$$\begin{aligned}\bar{F}(\xi,\eta,{}_{12}\boldsymbol{\lambda})&=[\alpha_{12}(\xi,x_2)]^2c_2(\xi,{}_{12}\boldsymbol{\lambda})\\&\quad-2\alpha_{12}(\xi,x_2)b_1(\xi,{}_{12}\boldsymbol{\lambda})b_2(\xi,{}_{12}\boldsymbol{\lambda})\\&\quad+[b_1(\xi,{}_{12}\boldsymbol{\lambda})]^2a_2(\xi)+[b_2(\xi,{}_{12}\boldsymbol{\lambda})]^2a_1(\xi)\\&\quad-a_1(\xi)a_2(\xi)c_2(\xi,{}_{12}\boldsymbol{\lambda})\\&=16m_1^2m_2^2[v(\xi)]^2c_2(\xi,{}_{12}\boldsymbol{\lambda})[x_2-f_+(\xi,{}_{12}\boldsymbol{\lambda})][x_2-f_-(\xi,{}_{12}\boldsymbol{\lambda})],\end{aligned} \tag{39}$$

where $f_\pm(\xi,{}_{12}\boldsymbol{\lambda})$ are defined by

$$\begin{aligned}\alpha_{12}(\xi,f_\pm(\xi,{}_{12}\boldsymbol{\lambda}))&=[c_2(\xi,{}_{12}\boldsymbol{\lambda})]^{-1}(b_1(\xi,{}_{12}\boldsymbol{\lambda})b_2(\xi,{}_{12}\boldsymbol{\lambda})\\&\quad\pm\{[b_1(\xi,{}_{12}\boldsymbol{\lambda})]^2-a_1(\xi)c_1(\xi,{}_{12}\boldsymbol{\lambda})\}^{1/2}\{[b_2(\xi,{}_{12}\boldsymbol{\lambda})]^2\\&\quad-a_2(\xi)c_2(\xi,{}_{12}\boldsymbol{\lambda})\}^{1/2})\end{aligned} \tag{40}$$

and the argument of Sec. 5 of I (or of VF) shows that

$$\begin{aligned}&(b_1(\xi,{}_{12}\boldsymbol{\lambda}))^2-a_1(\xi)c_1(\xi,{}_{12}\boldsymbol{\lambda})>0,\\&(b_2(\xi,{}_{12}\boldsymbol{\lambda}))^2-a_2(\xi)c_2(\xi,{}_{12}\boldsymbol{\lambda})>0\end{aligned} \tag{41}$$

for $\xi\geq h({}_{12}\boldsymbol{\lambda})$. The following points should now be noted. Firstly $\alpha_{12}(\xi,x_2)$ given in Eq. (31) is linear in x_2 so that the explicit expression for $f_\pm(\xi,{}_{12}\boldsymbol{\lambda})$ can easily be obtained from Eqs. (31) and (40). Secondly, as in Sec. 6 of I (or Sec. 5 of P) it is important to note that

$$c_1(\xi,{}_{12}\boldsymbol{\lambda})=c_2(\xi,{}_{12}\boldsymbol{\lambda})>0\quad\text{for }\xi>h({}_{12}\boldsymbol{\lambda})$$

in order to obtain Eqs. (37)–(40). Finally

$$\bar{F}(\xi,x_2,0)=64m_{-1}^2m_0^2m_1^2m_2^2[v(\xi)]^2F(\xi,x_2), \tag{42}$$

where $F(\xi,x_2)$, corresponding to the usual Mandelstam spectral function, is given in Eq. (B1) of Appendix B and $v(\xi)$ is given in Eq. (16).

The second Cauchy kernel now appears in the expression for $I_{n+2}(y_{ij})$ given by Eqs. (38), (37), and (19). To obtain the desired double spectral representation, it remains to reverse the order of the λ_i $(3\leq i\leq n)$ and ξ integrations and as well the λ_i and η integrations. The interchange of the λ_i $(3\leq i\leq n)$ and ξ integrations can be carried out by using the results of Secs. 3 and 4. Thus, using the expression for $I_{n+2}(y_{ij})$ in Eq. (19a), we have

$$\begin{aligned}I_{n+2}(y_{ij})&=\int_0^\infty\prod_{\substack{3\\ j\neq i}}^n d\lambda_j\Big(\prod_{\substack{3\\ j\neq i}}^n\lambda_j\Big)^{-1}\Big(\prod_{\substack{3\\ j\neq i}}^n\frac{\partial}{\partial m_j^2}\Big)\lim_{\epsilon_i\downarrow 0}\frac{\partial}{\partial m_i^2}\\&\quad\times\int_{h({}_{12}\lambda)|_{\lambda_i=\epsilon_i}}^\infty\frac{d\xi}{\xi-x_1}\int_{\epsilon_i}^{\lambda_{i+}(\xi,{}_{12i}\lambda)}\frac{d\lambda_i}{\lambda_i}\\&\quad\times\int_{f_+(\xi,{}_{12}\lambda)}^\infty\frac{d\eta}{\eta-x_2}\frac{8m_{-1}m_0m_1m_2v(\xi)}{[\bar{F}(\xi,\eta,{}_{12}\boldsymbol{\lambda})]^{1/2}}.\end{aligned} \tag{43}$$

The ξ and the other λ_j $(3\leq j\leq n, j\neq i)$ integrations can, of course, be reversed in a similar way, but it will be more convenient to interchange the order of the λ_i and η integrations before this is done.

6. STUDY OF $f_+(\xi,{}_{12}\lambda)$

To reverse the order of the λ_i and η integrations in Eq. (43), we need to examine the function $f_+(\xi,{}_{12}\boldsymbol{\lambda})$ for $0\leq\lambda_i\leq\lambda_{i+}(\xi,{}_{12i}\boldsymbol{\lambda})$ with fixed ${}_{12i}\boldsymbol{\lambda}\geq 0$, $\xi\geq h({}_{12i}\boldsymbol{\lambda})$, and $i\in\{3,\ldots,n\}$. First we study the behavior of $f_+(\xi,{}_{12}\boldsymbol{\lambda})$ as $\lambda_i\uparrow\lambda_{i+}(\xi,{}_{12i}\boldsymbol{\lambda})$. From Eq. (34) and the fact that $l_1({}_{12}\boldsymbol{\lambda})>0$ when Eq. (18) holds it follows that

$$b_1(\xi,{}_{12}\boldsymbol{\lambda})|_{\lambda_i=\lambda_{i+}(\xi,{}_{12i}\lambda)}<0. \tag{44}$$

Similarly

$$b_2(\xi,{}_{12}\boldsymbol{\lambda})|_{\lambda_i=\lambda_{i+}(\xi,{}_{12i}\lambda)}<0 \tag{45}$$

and hence from Eqs. (30) and (40) and the fact that

$$v(\xi)>0 \tag{46}$$

for $\xi\geq h({}_{12i}\boldsymbol{\lambda})$ (≥ 1) it follows that $f_+(\xi,{}_{12}\boldsymbol{\lambda})\to+\infty$ as $\lambda_i\uparrow\lambda_{i+}(\xi,{}_{12i}\boldsymbol{\lambda})$.

Next, from Eq. (40) we see that the derivative of $f_+(\xi,{}_{12}\boldsymbol{\lambda})$ with respect to λ_i is

$$\begin{aligned}f_{+\lambda_i}(\xi,{}_{12}\boldsymbol{\lambda})&=[4m_{-1}m_0v(\xi)]^{-1}[c_2(\xi,{}_{12}\boldsymbol{\lambda})]^{-2}\\&\quad\times(-b_1(\xi,{}_{12}\boldsymbol{\lambda})\{[b_2(\xi,{}_{12}\boldsymbol{\lambda})]^2-a_2(\xi)c_2(\xi,{}_{12}\boldsymbol{\lambda})\}^{1/2}\\&\quad-b_2(\xi,{}_{12}\boldsymbol{\lambda})\{[b_1(\xi,{}_{12}\boldsymbol{\lambda})]^2-a_1(\xi)c_1(\xi,{}_{12}\boldsymbol{\lambda})\}^{1/2})\\&\quad\times L_i(\xi,{}_{12}\boldsymbol{\lambda}),\end{aligned} \tag{47}$$

where

$$L_i(\xi,{}_{12}\boldsymbol{\lambda})=\frac{Q_i(\xi,{}_{12}\boldsymbol{\lambda})}{\sqrt{R_i(\xi,{}_{12}\boldsymbol{\lambda})}}+\frac{Q_i'(\xi,{}_{12}\boldsymbol{\lambda})}{\sqrt{R_i'(\xi,{}_{12}\boldsymbol{\lambda})}}, \tag{48}$$

$$\begin{aligned}Q_i(\xi,{}_{12}\boldsymbol{\lambda})&=\lambda_i(a_ib_1-\alpha_{1i}b_i)+(b_ib_1-\alpha_{1i}c_i),\\Q_i'(\xi,{}_{12}\boldsymbol{\lambda})&=\lambda_i(a_ib_2-\alpha_{2i}b_i)+(b_ib_2-\alpha_{2i}c_i),\\R_i(\xi,{}_{12}\boldsymbol{\lambda})&=\lambda_i^2(\alpha_{1i}^2-a_1a_i)+2\lambda_i(\alpha_{1i}b_1-a_1b_i)+b_1^2-a_1c_1,\\R_i'(\xi,{}_{12}\boldsymbol{\lambda})&=\lambda_i^2(\alpha_{2i}^2-a_2a_i)+2\lambda_i(\alpha_{2i}b_2-a_2b_i)+b_2^2-a_2c_2.\end{aligned} \tag{49}$$

In Eq. (49), a_j has been written for $a_j(\xi)$, c_j for $c_j(\xi,{}_{12i}\boldsymbol{\lambda})$, b_j for $b_j(\xi,{}_{12i}\boldsymbol{\lambda})$ $(j=1,2,i)$ and α_{li} for $\alpha_{li}(\xi,-y_{li})$ $(l=1,2)$.

The term in square brackets in Eq. (47) is always positive as can be seen as follows. From Eqs. (34), (29), and (18) and the fact that $b_1(\xi, {}_{12i}\lambda)$ is linear in ξ, we have

$$b_1(\xi, {}_{12i}\boldsymbol{\lambda}) < 0 \tag{50}$$

for all $\xi \geq h({}_{12i}\boldsymbol{\lambda})$, ${}_{12i}\boldsymbol{\lambda} \geq 0$. Then from Eqs. (44) and (50) and the fact that $b_1(\xi, {}_{12}\boldsymbol{\lambda})$ is linear in λ_i it follows that

$$b_1(\xi, {}_{12}\boldsymbol{\lambda}) < 0 \tag{51}$$

for $0 \leq \lambda_i \leq \lambda_{i+}(\xi, {}_{12i}\boldsymbol{\lambda})$ with fixed $\xi \geq h({}_{12i}\boldsymbol{\lambda})$, ${}_{12i}\boldsymbol{\lambda} \geq 0$. Similarly

$$b_2(\xi, {}_{12}\boldsymbol{\lambda}) < 0 \tag{52}$$

for $0 \leq \lambda_i \leq \lambda_{i+}(\xi, {}_{12i}\boldsymbol{\lambda})$.

The argument of Sec. 6 of P can now be used to show that, for fixed ${}_{12i}\boldsymbol{\lambda} \geq 0$, $\xi \geq h({}_{12i}\boldsymbol{\lambda})$, $f_+(\xi, {}_{12i}\boldsymbol{\lambda})$ is strictly increasing on $0 \leq \lambda_i \leq \lambda_{i+}(\xi, {}_{12i}\boldsymbol{\lambda})$ if and only if $L_i(\xi, {}_{12i}\boldsymbol{\lambda}) > 0$. That $L_i(\xi, {}_{12i}\boldsymbol{\lambda})$ is always positive for $\xi \geq h({}_{12i}\boldsymbol{\lambda})$, ${}_{12i}\boldsymbol{\lambda} \geq 0$ can be seen from Eq. (48) and Eqs. (B7) and (B8) of Appendix B. Next we shall find the inverse of the strictly increasing function $\eta = f_+(\xi, {}_{12}\boldsymbol{\lambda})$ for fixed $\xi \geq h({}_{12i}\boldsymbol{\lambda})$, ${}_{12i}\boldsymbol{\lambda} \geq 0$.

7. SOLUTIONS OF $\overline{F}(\xi, \eta, {}_{12}\lambda) = 0$

To obtain the inverse of $\eta = f_+(\xi, {}_{12}\boldsymbol{\lambda})$, we need to study the behavior of the zeros $\overline{F}(\xi, \eta, {}_{12}\boldsymbol{\lambda})$ for fixed ξ. η and fixed ${}_{12i}\boldsymbol{\lambda} \geq 0$. From Eqs. (29) and (31) we find that

$$F(\xi, \eta, {}_{12}\boldsymbol{\lambda}) = A_i(\xi, \eta)\lambda_i^2 + 2B_i(\xi, \eta, {}_{12i}\boldsymbol{\lambda})\lambda_i + C_i(\xi, \eta, {}_{12i}\boldsymbol{\lambda}), \tag{53}$$

where

$$\begin{aligned} A_i(\xi, \eta) &= a_i([\alpha_{12}(\xi, \eta)]^2 - a_1 a_2) + \alpha_{1i}^2 a_2 + \alpha_{2i}^2 a_1 \\ &\quad - 2\alpha_{12}(\xi, \eta)\alpha_{1i}\alpha_{2i}, \\ B_i(\xi, \eta, {}_{13i}\boldsymbol{\lambda}) &= b_i([\alpha_{12}(\xi, \eta)]^2 - a_1 a_2) + b_1\alpha_{1i} a_2 + b_2\alpha_{2i} a_1 \\ &\quad - \alpha_{12}(\xi, \eta) b_1 \alpha_{2i} - \alpha_{12}(\xi, \eta) b_2 \alpha_{1i}, \\ C_i(\xi, \eta, {}_{12i}\boldsymbol{\lambda}) &= c_i([\alpha_{12}(\xi, \eta)]^2 - a_1 a_2) + b_1^2 a_2 + b_2^2 a_1 \\ &\quad - 2\alpha_{12}(\xi, \eta) b_1 b_2. \end{aligned} \tag{54}$$

The functions $A_i(\xi, \eta)$ and $B_i(\xi, \eta, {}_{12i}\boldsymbol{\lambda})$ in fact depend on m_i^2 whereas $C_i(\xi, \eta, {}_{12i}\boldsymbol{\lambda})$ does not. The abbreviations described after Eq. (49) have again been used.

As in Sec. 7 of P, we find that the discriminant of the quadratic function of λ_i in Eq. (53) is

$$\begin{aligned} &[B_i(\xi, \eta, {}_{12i}\boldsymbol{\lambda})]^2 - A_i(\xi, \eta)C_i(\xi, \eta, {}_{12i}\boldsymbol{\lambda}) \\ &\quad = \{[\alpha_{12}(\xi, \eta)]^2 - a_1 a_2\}\overline{P}_i(\xi, \eta, {}_{12i}\boldsymbol{\lambda}), \end{aligned} \tag{55}$$

where

$$\begin{aligned} \overline{P}_i(\xi, \eta, {}_{12i}\boldsymbol{\lambda}) &= 16 m_1^2 m_2^2 [v(\xi)]^2 (b_i^2 - a_i c_i) \\ &\quad \times [\eta - p_{i+}(\xi, {}_{12i}\boldsymbol{\lambda})][\eta - p_{i-}(\xi, {}_{12i}\boldsymbol{\lambda})]. \end{aligned} \tag{56}$$

The functions $p_{i\pm}(\xi, {}_{12i}\boldsymbol{\lambda})$ are given by

$$\begin{aligned} \alpha_{12}(\xi, p_{i\pm}(\xi, {}_{12i}\boldsymbol{\lambda})) &= (b_i^2 - a_i c_i)^{-1}\{b_i(b_1\alpha_{2i} + b_2\alpha_{1i}) - a_i b_1 b_2 \\ &\quad - c_i\alpha_{1i}\alpha_{2i} \pm [D_i(\xi, {}_{12i}\boldsymbol{\lambda}) D_i'(\xi, {}_{12i}\boldsymbol{\lambda})]^{1/2}\}, \end{aligned} \tag{57}$$

where

$$D_i(\xi, {}_{12i}\boldsymbol{\lambda}) = c_i(\alpha_{1i}^2 - a_1 a_i) + b_1^2 a_i + b_i^2 a_1 - 2\alpha_{1i} b_1 b_i,$$

$$D_i'(\xi, {}_{12i}\boldsymbol{\lambda}) = c_i(\alpha_{2i}^2 - a_2 a_i) + b_2^2 a_i + b_i^2 a_2 - 2\alpha_{2i} b_2 b_i, \tag{58}$$

and $\alpha_{12}(\xi, \eta)$ given in Eq. (31) is linear in η.

The discriminant in Eq. (55) is always nonnegative for $\eta \geq f_+(\xi, {}_{12i}\boldsymbol{\lambda})$, $\xi \geq h({}_{12i}\boldsymbol{\lambda})$, ${}_{12i}\boldsymbol{\lambda} \geq 0$, since the inverse of $\eta = f_+(\xi, {}_{12}\boldsymbol{\lambda})$ is real. That it is in fact positive can be seen as follows.

In Sec. 7 of P we showed that

$$\{[\alpha_{12}(\xi, \eta)]^2 - a_1(\xi)a_2(\xi)\} > 0 \tag{59}$$

for all $\xi \geq h(0) = 1$, $\eta \geq f_+(\xi, 0)$. Since for ${}_{12i}\boldsymbol{\lambda} \geq 0$, $h({}_{12i}\boldsymbol{\lambda}) \geq 1$ and for $\xi \geq h({}_{12i}\boldsymbol{\lambda})$, $f_+(\xi, {}_{12i}\boldsymbol{\lambda}) \geq f_+(\xi, 0)$ Eq. (59) holds in particular for $\xi \geq h({}_{12i}\boldsymbol{\lambda})$, $\eta \geq f_+(\xi, {}_{12i}\boldsymbol{\lambda})$. Further, it is shown in Eq. (B3) of Appendix B that $\overline{P}(\xi, \eta, {}_{12i}\boldsymbol{\lambda})$ is positive for $\xi \geq h({}_{12i}\boldsymbol{\lambda})$, $\eta \geq f_+(\xi, {}_{12i}\boldsymbol{\lambda})$.

The two real solutions of

$$\overline{F}(\xi, \eta, {}_{12}\boldsymbol{\lambda}) = 0 \tag{60}$$

are

$$\begin{aligned} &\lambda_{i\,{a \atop b}}(\xi, \eta, {}_{12i}\boldsymbol{\lambda}) \\ &\quad = [A_i(\xi, \eta)]^{-1}(-B_i(\xi, \eta, {}_{12i}\boldsymbol{\lambda}) \mp \{[B_i(\xi, \eta, {}_{12i}\boldsymbol{\lambda})]^2 \\ &\qquad - A_i(\xi, \eta)C_i(\xi, \eta, {}_{12i}\boldsymbol{\lambda})\}^{1/2}). \end{aligned} \tag{61}$$

From Eqs. (54), (48), (B7), (B8), (51), and (52) we see that

$$\begin{aligned} &B_i(\xi, f_+(\xi, {}_{12i}\boldsymbol{\lambda}), {}_{12i}\boldsymbol{\lambda}) \\ &= c_i^{-2}[b_1(R_i'(\xi, {}_{12i}\boldsymbol{\lambda}))^{-1/2} + b_2(R_i(\xi, {}_{12i}\boldsymbol{\lambda}))^{-1/2}]L_i(\xi, {}_{12i}\boldsymbol{\lambda}) < 0 \end{aligned} \tag{62}$$

so that

$$\lambda_{ia}(\xi, f_+(\xi, {}_{12i}\boldsymbol{\lambda}), {}_{12i}\boldsymbol{\lambda}) = 0 \neq \lambda_{ib}(\xi, f_+(\xi, {}_{12i}\boldsymbol{\lambda}), {}_{12i}\boldsymbol{\lambda}). \tag{63}$$

Also as $\eta \to +\infty$

$$\lambda_{i\,{a \atop b}}(\xi, \eta, {}_{12i}\boldsymbol{\lambda}) \sim \lambda_{i\pm}(\xi, {}_{12i}\boldsymbol{\lambda}),$$

where $\lambda_{i\pm}(\xi, {}_{12i}\boldsymbol{\lambda})$ are given in Eq. (33). Thus $\lambda_{ia}(\xi, \eta, {}_{12i}\boldsymbol{\lambda})$ is the inverse of the strictly increasing function $f_+(\xi, {}_{12}\boldsymbol{\lambda})$ on $0 \leq \lambda_i \leq {}_{i+}(\xi, {}_{12i}\boldsymbol{\lambda})$, where ξ is fixed such that $\xi \geq h({}_{12i}\boldsymbol{\lambda})$ and ${}_{12i}\boldsymbol{\lambda} \geq 0$. The function $\lambda_{ia}(\xi, \eta, {}_{12i}\boldsymbol{\lambda})$ increases strictly from 0 to $\lambda_{i+}(\xi, {}_{12i}\boldsymbol{\lambda})$ as η increases from $f_+(\xi, {}_{12i}\boldsymbol{\lambda})$ to $+\infty$.

8. DOUBLE SPECTRAL REPRESENTATION OF THE k^{th} ORDER SINGLE LOOP AMPLITUDE

The order of the λ_i and η integrations in Eq. (43) can now be interchanged, and we find, on taking $i = 3$, that

$$\begin{aligned} I_{n+2}(y_{ij}) &= \int_0^\infty \prod_{j=4}^n d\lambda_j \left(\prod_{j=4}^n \lambda_j\right)^{-1} \left(\prod_{j=4}^n \frac{\partial}{\partial m_j^2}\right) \lim_{\epsilon_3 \downarrow 0} \frac{\partial}{\partial m_3^2} \\ &\quad \times \int_{h({}_{12}\boldsymbol{\lambda})|_{\lambda_3=\epsilon_3}}^\infty \frac{d\xi}{\xi - x_1} \int_{f_+(\xi, {}_{12}\boldsymbol{\lambda})|_{\lambda_3=\epsilon_3}}^\infty \frac{d\eta}{\eta - x_2} \\ &\quad \times \int_{\epsilon_3}^{\lambda_{3a}(\xi, \eta, {}_{123}\boldsymbol{\lambda})} \frac{d\lambda_3}{\lambda_3} \frac{8 m_{-1} m_0 m_1 m_2 v(\xi)}{[\overline{F}(\xi, \eta, {}_{12}\boldsymbol{\lambda})]^{1/2}}. \end{aligned} \tag{64}$$

The results of Appendix C can be used to interchange the order of $\lim_{\epsilon_3 \downarrow 0} \partial/\partial m_3^2$ and integration with respect to ξ and η. Defining the operators

$$O_j \equiv \lim_{\epsilon_j \downarrow 0} \frac{\partial}{\partial m_j^2} \int_{\epsilon_j}^{\lambda_{ja}(\xi,\eta,{}_{12\cdots j}\lambda)} \frac{d\lambda_j}{\lambda_j} \quad (3 \leqslant j \leqslant n), \tag{65}$$

we find that

$$I_{n+2}(y_{ij}) = \int_0^\infty \prod_{j=4}^n d\lambda_j \left(\prod_{j=4}^n \lambda_j\right)^{-1} \left(\prod_{j=4}^n \frac{\partial}{\partial m_j^2}\right) \times \int_{h({}_{123}\lambda)}^\infty \frac{d\xi}{\xi - x_1} \int_{f_+(\xi,{}_{123}\lambda)}^\infty \frac{d\eta}{\eta - x_2} O_3 \times \frac{8m_{-1}m_0m_1m_2v(\xi)}{[\overline{F}(\xi,\eta,{}_{12}\lambda)]^{1/2}} . \tag{66}$$

Note that the limit $\epsilon_3 \downarrow 0$ is now inside the ξ and η integrations and in fact $O_3[\overline{F}(\xi,\eta,{}_{12}\lambda)]^{-1/2}$ can be evaluated as in Eq. (C1).

With the expression for $I_{n+2}(y_{ij})$ given in Eqs. (19b), (37), and (38), we can use the results of Secs. 3 and 4 to reverse the order of the λ_4 and ξ integrations and the results of Secs. 6 and 7 to interchange the order of the λ_4 and η integrations. The order of $\lim_{\epsilon_4 \downarrow 0} \partial/\partial m_4^2$ and the integration with respect to ξ and η can then be interchanged by using the results of Appendix C. Repeating the process, we find that

$$I_{n+2}(y_{ij}) = \int_1^\infty \frac{d\xi}{\xi - x_1} \int_{f_+(\xi)}^\infty \frac{d\eta}{\eta - x_2} O_n O_{n-1} \cdots O_3 \times \frac{8m_{-1}m_0m_1m_2}{[\overline{F}(\xi,\eta,{}_{12}\lambda)]^{1/2}} \tag{67}$$

giving the required double spectral representation with Mandelstam boundary for the single loop amplitude of order $k=n+2$ (>4). In Eq. (67) $f_+(\xi) \equiv f_+(\xi, \mathbf{0})$ is given in Eq. (B2) and $h(\mathbf{0})=1$.

From Eq. (42) it can be seen that with all the O_j operators missing $(3 \leqslant j \leqslant n)$, ${}_{12}\boldsymbol{\lambda} \to \mathbf{0}$ and $n=2$, Eq. (67) is just the double spectral representation obtained for the box diagram amplitude in Eq. (I-97). Also with only O_3 appearing in Eq. (67), ${}_{12}\boldsymbol{\lambda} \to \lambda_3$ and $n=3$, we see from Eq. (C1) that Eq. (67) is just the double spectral representation for the pentagon diagram amplitude given in Eq. (P-64) and hence in Eq. (P-65). While it is much more tedious to evaluate some of the higher order spectral functions from Eq. (67), such calculations would provide interesting checks on the Cutkosky rules which, to the best of my knowledge, have only been checked for the lowest order amplitudes (see, for example, also Chap. 4, Sec. 3 of Ref. 4).

It remains to show that the conditions in Eq. (18) under which the double spectral representation for the kth order single-loop amplitude has been proved, can, for sufficiently large internal masses, be satisfied for finite physical values of the kinematic invariants and external masses. Further we shall discuss how one can obtain a representation for the amplitude when the conditions in Eq. (18) are, at least to some extent, relaxed. We begin by considering the box diagram amplitude. For a particular channel reaction, the equations determining the region in which the kinematic invariants take on physical values (given, for example, in Ref. 8) depend only on the kinematic invariants and external masses. Thus, for finite physical values of the kinetic invariants and external masses, Eq. (18) can be satisfied provided the internal masses are sufficiently large. To obtain a representation for more general values of the y_{ij}, one can do an analytic continuation in these variables as was done in II. While, as found there, it is rather laborious to obtain a representation for almost all real kinematic invariants and for all possible mass configurations involving stable external particles, the continuation in the kinematic invariants can readily be carried out when the mass variables satisfy Eq. (18). Then, the representation from which the continuation is started is the double spectral representation which contains x_1 and x_2 only in the Cauchy kernels. Thus for the x_1 channel reaction, which in Fig. 2 corresponds to $n=2$, p_{-11} (rather than $-p_{-11}$) and p_{01} being incoming and p_{-12} and p_{02} (rather than their negatives) being outgoing 4-momenta, the amplitude has, for physical invariants, the representation given in Eq. (I-97) and in Eq. (67) with $x_1 \to x_1 + i \cdot 0$ [and as described earlier, with the O_j operators missing $(3 \leqslant j \leqslant n)$ and ${}_{12}\boldsymbol{\lambda} \to 0$]. In fact, as shown in Secs. 6 and 8 of I, Eq. (I-97) holds under slightly more general conditions on the mass variables than those given in Eq. (18). In a similar way, for a general kth order single-loop amplitude one would expect Eq. (67) to be valid under more general conditions than those given in Eq. (18); that continuation in x_1 and x_2 can easily be carried out is of course obvious.

For the pentagon diagram amplitude, the equations which, for a particular channel reaction, define the region in which the kinematic invariants take on physical values again depend only on the kinematic invariants and external masses (see, for example, Section 4.3 of Ref. 4). Thus Eq. (18) can be satisfied for finite physical values of the kinematic invariants and external masses provided the internal masses are sufficiently large. It should, however, be noted that, for pentagon diagram amplitudes associated with most physically interesting production reactions involving hadrons, for example $\pi N \to \pi\pi N$, the lowest mass intermediate particles which can be exchanged are such that complex singularities appear on the physical sheet,[26] even for the smallest possible physical values of the invariants, causing a breakdown of the double (and even single) dispersion relations in x_1 and x_2. Hence one would not expect dispersion relations, over real contours, in these variables to be valid for the total production amplitudes. In fact, for the reaction $\pi N \to \pi\pi N$, complex singularities are also produced by lower order contracted diagrams.[27] To obtain a representation for the pentagon diagram amplitudes when the internal masses are small, one might attempt to generalize the method of continuation used in II. On the basis of the work of Cook and Tarski,[26] it seems that at least a numerical study of the motion of the singularities for specific processes is feasible.

Finally, for the kth order single-loop amplitude where $k \geqslant 6$, we mentioned in Sec. 2 that the kinematic invariants defined in Eqs. (2) or (3) are not independent but satisfy algebraic constraints.[22] These constraints, however, involve only the kinematic invariants and the external masses. Further the equations which, for a particular channel reaction, define the region in which the kinematic invariants take on physical values again depend only on the kinematic invariants and the external masses.[4] Thus, for finite physical values of the kinematic invariants and external masses, the conditions in Eq. (18), under which the double spectral representation was proved, can be satisfied provided the internal masses are sufficiently large. In fact, as mentioned earlier, Eq. (67) is expected to hold under slightly more general conditions than those given in Eq. (18). The double (and even single) dispersion relations in x_1 and x_2 will of course break down for sufficiently small internal masses. In such cases the method of analytic continuation unfortunately seems of little use for finding a representation of the amplitude, simply because of the increased number of singularities and the more complicated nature of the spectral function in Eq. (67).

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APPENDIX A

In this appendix we outline the method of induction used to obtain Eqs. (10)—(17). It was shown in Sec. 2 of P that these equations hold for the case $n=3$ (where $n=k-2$ and k is the number of vertices of the single loop amplitude). We suppose that Eqs. (6)—(17), with the replacement $n\to l$, hold for all $3\leq l\leq n-1$ and show that they are then valid as well for $l=n$. The steps in the proof are as follows.

(1) In Eqs. (7)—(9) make the change of variables $\zeta=(1-\alpha_n)^{-1}$ and then $x_i=\zeta\alpha_i$ ($i\neq 1$, $-1\leq i\leq n-1$). The Jacobian of the transformation is $\zeta^{-(n+1)}$ and in terms of the new variables

$$D_n(\alpha_{-1},\alpha_0,\alpha_2,\ldots,\alpha_n)\equiv m_n^2(\zeta-1)^2\zeta^{-2}+\sum_{\substack{-1\\ i\neq 1}}^{n-1}2m_im_ny_{in}(\zeta-1)\zeta^{-2}x_i+2m_1m_ny_{1n}\Big(1-\sum_{\substack{-1\\ i\neq 1}}^{n-1}x_i\Big)(\zeta-1)\zeta^{-2}+\zeta^{-2}D_{n-1}(x_{-1},x_0,x_2,\ldots x_{n-1}).\tag{A1}$$

(2) Make the change of variables given in Eqs. (10) and (11) with the replacements $n\to n-1$, $\alpha_i\to x_i$. With these replacements, the Jacobian of the transformation is, by assumption, given in Eq. (12) and using Eq. (13), again with the above replacements, we find that

$$\begin{aligned}D_n(\alpha_{-1},\alpha_0,\alpha_2,\ldots,\alpha_n)\equiv\zeta^{-2}\Big[&m_n^2(\zeta-1)^2+\sum_{i=2}^{n-1}2m_im_ny_{in}\lambda_i(\zeta-1)\Big(1+\sum_{j=1}^{n-1}\lambda_j\Big)^{-1}\\&+2m_0m_ny_{0n}\nu^{-1}(\zeta-1)\Big(1+\sum_{j=1}^{n-1}\lambda_j\Big)^{-1}\\&+2m_{-1}m_ny_{-1n}\nu^{-1}(\nu-1)(\zeta-1)\Big(1+\sum_{j=1}^{n-1}\lambda_j\Big)^{-1}\\&+2m_1m_ny_{1n}\lambda_1(\zeta-1)\Big(1+\sum_{j=1}^{n-1}\lambda_j\Big)^{-1}\\&+\nu^{-1}\Big(1+\sum_{j=1}^{n-1}\lambda_j\Big)^{-2}[(\nu-1)\phi(\lambda_1,\ldots,\lambda_{n-1},0)\\&+\psi(\lambda_1,\ldots,\lambda_{n-1},0)\\&-\nu^{-1}(\nu-1)v(x_1)]\Big].\end{aligned}\tag{A2}$$

(3) Make the change of variable

$$\lambda_n=(\zeta-1)\Big(1+\sum_{j=1}^{n-1}\lambda_j\Big)$$

with the inverse

$$\zeta=\Big(1+\sum_{j=1}^{n}\lambda_j\Big)\Big(1+\sum_{j=1}^{n-1}\lambda_j\Big)^{-1}.$$

The Jacobian of the transformation is $(1+\sum_{j=1}^{n-1}\lambda_j)^{-1}$ and we find that $D_n(\alpha_{-1},\alpha_0,\alpha_2,\ldots,\alpha_n)\equiv\Delta_n(\nu,\lambda_1,\ldots,\lambda_n)$ as given in Eqs. (13)—(16).

Compounding the transformations and Jacobians in steps (1)—(3) we find that the resultant transformation is just that given in Eq. (10) with the inverse as in Eq. (11). Further the resultant Jacobian is as given in Eq. (12). It is then readily seen that the new region of integration and expression for $I_{n+2}(y_{ij})$ are as given in Eq. (17).

APPENDIX B

We collect here a number of results involving the various functions needed in the main body of the paper. It is assumed throughout that Eq. (18) holds. From Eqs. (42) and (39)

$$F(\xi,x_2)=(\xi^2-1)[x_2-f_+(\xi)][x_2-f_-(\xi)],\tag{B1}$$

where from Eqs. (40) and (31)

$$\begin{aligned}f_\pm(\xi)\equiv f_\pm(\xi,0)=(\xi^2-1)^{-1}[&(\xi-1)(y_{-11}y_{02}+y_{01}y_{-12})\\&+(y_{-11}+y_{01})(y_{-12}+y_{02})\\&\pm(\xi^2+2y_{-11}y_{01}\xi+y_{-11}^2+y_{01}^2-1)^{1/2}\\&\times(\xi^2+2y_{-12}y_{02}\xi+y_{-12}^2+y_{02}^2-1)^{1/2}].\end{aligned}\tag{B2}$$

The above functions, with a relabelling of variables, were also defined in Eqs. (I-11) and (I-12) and their properties were discussed in detail in Sec. 8 of I and in Sec. 4 of II.

Next we show that, for fixed $i\in\{3,\ldots,n\}$,

$$\bar{P}_i(\xi,\eta,{}_{12i}\lambda)>0\tag{B3}$$

for all $\xi\geq h({}_{12i}\lambda)$, $\eta\geq f_+(\xi,{}_{12i}\lambda)$, where $\bar{P}_i(\xi,\eta,{}_{12i}\lambda)$ is defined in Eqs. (56), (57), and (58) and the abbreviations described after Eq. (49) have again been used. In the same way as Eq. (41) was established, we find that

$$b_i^2-a_ic_i>0\tag{B4}$$

for $\xi\geq h({}_{12i}\lambda)$ and hence Eq. (B3) will hold if $p_{i\pm}(\xi,{}_{12i}\lambda)$, given in Eq. (57), are either complex conjugates or if

$$p_{i-}(\xi,{}_{12i}\lambda)\leq p_{i+}(\xi,{}_{12i}\lambda)<f_+(\xi,{}_{12i}\lambda).\tag{B5}$$

That both alternatives are in fact possible can be seen from Appendix A of P. Thus we have the following cases to consider.

(i) $D_i(\xi,{}_{12i}\lambda)D_i'(\xi,{}_{12i}\lambda)<0$. Then $p_{i\pm}(\xi,{}_{12i}\lambda)$ are complex conjugates and Eq. (B3) holds.

(ii) $D_i(\xi, {}_{12i}\lambda) \geq 0$, $D_i'(\xi, {}_{12i}\lambda) \geq 0$. From Eqs. (40), (57), (58), (49) and the fact that $c_1 = c_2 = c_i$, we have

$$\begin{aligned}
&\alpha_{12}(\xi, p_{i\pm}(\xi, {}_{12i}\lambda)) - \alpha_{12}(\xi, f_+(\xi, {}_{12i}\lambda)) \\
&= (b_i^2 - a_i c_i)^{-1} c_2^{-1}[-Q_i(\xi, {}_{12i}\lambda)Q_i'(\xi, {}_{12i}\lambda) \\
&\quad \pm (\{[Q_i(\xi, {}_{12i}\lambda)]^2 - (b_i^2 - a_i c_i)(b_1^2 - a_1 c_1)\} \\
&\quad \times \{[Q_i'(\xi, {}_{12i}\lambda)]^2 - (b_i^2 - a_i c_i)(b_2^2 - a_2 c_2)\})^{1/2} \\
&\quad - (b_i^2 - a_i c_i)(b_1^2 - a_1 c_1)^{1/2}(b_2^2 - a_2 c_2)^{1/2}]. \qquad \text{(B6)}
\end{aligned}$$

Now from Eqs. (49), (31), (30), (21) and (27) it can be shown that

$$\begin{aligned}
&Q_i(\xi, {}_{12i}\lambda) \\
&= 4v(\xi)\{y_{1i}c_1 + 2(q_1 q_i' + q_1' q_i)[v(\xi) - r_1 - r_1'] + 4q_i' q_1' r_1 \\
&\quad + 4q_i q_1 r_1'\} > 0 \qquad \text{(B7)}
\end{aligned}$$

for $\xi \geq h({}_{12i}\lambda)$. Here we have used the abbreviation q_j for $q_j({}_{12i}\lambda)$ $(j=1, i)$ etc. It is also important to note that $r_1 = r_i$ $(= r_2)$ and $c_1 = c_i$ $(= c_2)$. Similarly

$$Q_i'(\xi, {}_{12i}\lambda) > 0 \qquad \text{(B8)}$$

for $\xi \geq h({}_{12i}\lambda)$. Defining

$$\cosh\kappa_1 = (b_1^2 - a_1 c_1)^{-1/2}(b_i^2 - a_i c_i)^{-1/2} Q_i(\xi, {}_{12i}\lambda), \qquad \text{(B9)}$$

$$\cosh\kappa_2 = (b_2^2 - a_2 c_2)^{-1/2}(b_i^2 - a_i c_i)^{-1/2} Q_i'(\xi, {}_{12i}\lambda), \qquad \text{(B10)}$$

we can then use the method of Appendix A of P and Eq. (31) to show that Eq. (B5) and hence Eq. (B3) hold.

(iii) $D_i(\xi, {}_{12i}\lambda) < 0$, $D_i'(\xi, {}_{12i}\lambda) < 0$. In this case we define $\cos\phi_1$ (resp. $\cos\phi_2$) by the right-hand side of Eq. (B9) [resp. (B10)] and again Eq. (B5) and hence Eq. (B3) hold.

APPENDIX C

In this appendix we outline the method of interchanging the order of $\lim_{\epsilon_j \downarrow 0} \partial/\partial m_j^2$ $(3 \leq j \leq n)$ and the integrations with respect to ξ and η, which is needed to obtain Eq. (67) in Sec. 8. The method is very similar to that described in Secs. 6, 7 and Appendix B of I and in Sec. 8 of P. From Eqs. (22), (14), and (15) we find that $(\partial/\partial m_j^2)h(\lambda) = O(\lambda_j)$ as $\lambda_j \downarrow 0$ and from Eqs. (40), (31), and (30) $(\partial/\partial m_j^2) f_+(\xi, {}_{12}\lambda) = O(\lambda_j)$ as $\lambda_j \downarrow 0$. Further, as noted after Eq. (54), $A_j(\xi, \eta)$ and $B_j(\xi, \eta, {}_{12j}\lambda)$ depend on m_j^2 whereas $C_j(\xi, \eta, {}_{12j}\lambda)$ $[\equiv \overline{F}(\xi, \eta, {}_{12j}\lambda)]$ does not. Thus the argument of Sec. 8 of P can be used to show that $I_{n+2}(y_{ij})$ given in Eq. (64) can also be written as in Eq. (66). In fact, it follows from Sec. 8 of P that, with O_3 defined in Eq. (65),

$$\begin{aligned}
&O_3[\overline{F}(\xi, \eta, {}_{12}\lambda)]^{-1/2} \\
&= \frac{-\frac{1}{2}(\partial/\partial m_3^2)\{[B_3(\xi, \eta, {}_{123}\lambda)]^2 - A_3(\xi, \eta)C_3(\xi, \eta, {}_{123}\lambda)\}}{[\overline{F}(\xi, \eta, {}_{123}\lambda)]^{1/2}\{[B_3(\xi, \eta, {}_{123}\lambda)]^2 - A_3(\xi, \eta)C_3(\xi, \eta, {}_{123}\lambda)\}}. \qquad \text{(C1)}
\end{aligned}$$

The term in square brackets in Eq. (C1) never vanishes in the region of integration in Eq. (66) since, as can be seen from Eqs. (55), (56), (59), and (B3),

$$\{[B_j(\xi, \eta, {}_{12j}\lambda)]^2 - A_j(\xi, \eta)C_j(\xi, \eta, {}_{12j}\lambda)\} > 0 \qquad \text{(C2)}$$

for all $\xi \geq h({}_{12j}\lambda)$, $\eta \geq f_+(\xi, {}_{12j}\lambda)$, ${}_{12j}\lambda \geq 0$. In fact, $O_3[\overline{F}(\xi, \eta, {}_{12}\lambda)]^{-1/2}$ can be majorized by $M_3[\overline{F}(\xi, \eta, {}_{123}\lambda)]^{-1/2}$, where M_3 is a positive constant. To repeat the process of interchanging the order of $\lim_{\epsilon_j \downarrow 0} \partial/\partial m_j^2$ $(3 < j \leq n)$ and the ξ and η integrations, it is necessary in addition to use the theorem given, for example, in Section 225 of Hobson[28] and the fact that $O_j \ldots O_3[\overline{F}(\xi, \eta, {}_{12}\lambda)]^{-1/2}$ can be majorized by $M_j[\overline{F}(\xi, \eta, {}_{123\cdots j}\lambda)]^{-1/2}$, where M_j is a positive constant.

[1]N. N. Bogoliubov, B.V. Medvedev, and M.K. Polivanov, Fortschr. Phys. 6, 159 (1959); H.J. Bremermann, R. Oehme, and J.G. Taylor, Phys. Rev. 109, 2178 (1958); H. Lehmann, Nuovo Cimento 10, 579 (1958).
[2]R.J. Eden, P.V. Landshoff, D.I. Olive, and J.C. Polkinghorne, *The Analytic S-Matrix* (Cambridge U.P., Cambridge, 1966).
[3]N. Nakanishi, *Graph Theory and Feynman Integrals* (Gordon and Breach, New York, 1971).
[4]I. T. Todorov, *Analytic Properties of Feynman Diagrams in Quantum Field Theory* (Pergamon, New York, 1971).
[5]R. Karplus, C.M. Sommerfield, and E.H. Wichmann, Phys. Rev. 111, 1187 (1958).
[6]Y. Nambu, Nuovo Cimento 9, 610 (1958).
[7]R. Oehme, Phys. Rev. 111, 1430 (1958).
[8]S. Mandelstam, Phys. Rev. 115, 1741, 1752 (1959).
[9]S. Mandelstam, Phys. Rev. 112, 1344 (1959).
[10]L. D. Landau, Nucl. Phys. 13, 181 (1959).
[11]R.E. Cutkosky, J. Math. Phys. 1, 429 (1960).
[12]P.V. Landshoff and S.B. Treiman, Phys. Rev. 127, 649 (1962); J.J. Kubis and J.L. Gammel, Phys. Rev. 172, 1664 (1968).
[13]R.E. Norton, Phys. Rev. 135, B1381 (1964).
[14]R.E. Cutkosky, Rev. Mod. Phys. 33, 448 (1961).
[15]R.C. Hwa and V.L. Teplitz, *Homology and Feynman Integrals* (Benjamin, New York, 1966).
[16]D. Fotiadi, M. Froissart, J. Lascoux, and F. Pham, "Analytic Properties of some integrals over a complex manifold," Paris (1963) (preprint); Topology 4, 159 (1965) (reprinted in Ref. 15).
[17]D. Fotiadi and F. Pham, pp. 192—244 of Ref. 15; G. Ponzano, T. Regge, E.R. Speer, and M.J. Westwater, Commun. Math. Phys. 18, 1 (1970); further references are given in Refs. 4 and 15.
[18]P. Federbush, J. Math. Phys. 6, 274 (1965); J.M. Westwater, Helv. Phys. Acta 40, 596 (1967).
[19]J.S. Frederiksen and W.S. Woolcock, Nucl. Phys. B 28, 605 (1971).
[20]J.S. Frederiksen and W.S. Woolcock, Ann. Phys. (N.Y.) 75, 503 (1973).
[21]J.S. Frederiksen, "Spectral representation of the pentagon diagram amplitude," to be published in J. Math. Phys.
[22]V.E. Asribekov, Zh. Eksp. Teor. Fiz. 42 565 (1962) [Sov. Phys. JETP 15, 394 (1962)].
[23]S.S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row, New York, 1961).
[24]J. Tarski, J. Math. Phys. 1, 149 (1960).
[25]J.S. Frederiksen and W.S. Woolcock, Ann. Phys. (N.Y.) 80, 86, (1973).
[26]L. F. Cook and J. Tarski, J. Math. Phys. 3, 1 (1962).
[27]D. Bessis and F. Pham, J. Math. Phys. 4, 1253 (1963).
[28]E.W. Hobson, *The Theory of Functions of a Real Variable and the Theory of Fourier's Series* (Dover, New York, 1957).